

Chapter 15: Understanding Phase Diagrams

Purpose of this Lab:

This lab introduces a very important concept in the analysis of equilibration processes. This material is somewhat difficult, so read and re-read this handout carefully. Read with a pen in your hand. Underline important phrases and jot down notes and questions. Be an *active reader*.

Previously:

Chapter 14 focused on:

- 1) Finding the **initial solution** both by pencil and paper and two ways by computer.
- 2) Analyzing the **equilibration process** by varying a key parameter of the difference equation which described the process and exploring the effect of changes in the parameter (θ) on three graphs:
 - a) Profit over time
 - b) Price over time
 - c) Profit as a function of Price
- 3) **Comparative Statics** by exploring how P_e (equilibrium price) varies with $C(Q)$.

Some important points we would like to stress include:

- 1) No one picks P_e . Agents make decisions (based upon the signal) that all together generate a response which then affects the signal.
- 2) P_e does not necessarily equal P^* . Judging a particular equilibrium outcome requires additional work. You cannot simply conclude that equilibrium is optimal.
- 3) Try to carefully match the optimization and equilibrium frameworks in order to clearly see the "big picture." There is no constrained/unconstrained fork in the equilibrium analysis, but there is an "equilibration process" component in Step 2) Find the Initial Solution.
- 4) Learn how to use Solver to generate the equilibrium solution from mathematical equation(s). You don't max or min, but set the equilibrium condition = 0.
- 5) The initial solution may be described as an intersection (as in supply and demand analysis), but the intersection point is NOT THE CAUSE of equilibrium.

Today:

Today, we will focus on **Step 2) Finding the Equilibrium Solution**. We will analyze the equilibration process by using a new graphical procedure called a **phase diagram**.

Somereview:

Let's review the part of Chapter 14 that described the profit equilibration model in order to set the stage for today's discussion of the equilibration process with phase diagrams:

The equilibration process tells us how the endogenous variables in the system respond to the forces in the model. We will carefully examine one possible equilibration process. Let's suppose that P at any point in time is determined according to the following **difference equation**:

$$P(t+1) = P(t) + \Delta P(t+1) \quad \text{where} \quad \Delta P(t+1) = \theta \cdot \pi(t)$$

Let's figure out what this says in plain English. P stands for price, and we've added a label that tells you what period that price belongs to. P(t) stands for price at time t; and P(t+1) stands for price in the NEXT period after t, time period t+1. The labels "t" or "t+1" allow us to make general statements regarding time. For *any* value of "t," "t+1" is the NEXT period and "t+8" is eight periods later. The difference equation above tells you how P(t) and P(t+1) are related; or in other words, how P changes over time.

You read the difference equation like this:

The price in time period t+1 is equal to the price in time period t (or what the price was the period previous to t+1) plus the change in the price from time period t to t+1
where the change in the price from time period t to time period t+1 is equal to theta times the level of profits in time period t.

Basically, the equation above says that the change (Δ) in price from the current period to the next period is some proportion (theta) of the level of profits in the current period. Theta is some exogenous variable whose value depends on the system you're looking at. Different values of theta will generate different kinds of equilibration.

In Chapter 14, we gave a specific example of how prices would change, starting from an initial price of 18 and assuming a theta of -0.015. You may wish to look back at that discussion, which is on pp. 7-8. The price bounced above and below equilibrium and eventually settled down to $P_e=10$. We called this oscillatory convergence.

We have seen that there are different types of equilibration processes. Economists are especially interested in determining the following 3 characteristics of an equilibrium solution:

- 1) Is it a **Stable** Equilibrium (Convergent Equilibration) or **Unstable** Equilibrium (Divergent Equilibration)?
- 2) Is it an **Oscillatory** or **Non-Oscillatory** Equilibration Process?
- 3) Is it a **Slow** or **Fast** Equilibration Process?

These different characteristics can be depicted by graphs of the endogenous variable over time. That was the point of the questions on the last page of Chapter 14. We give the answers below:

Value of Theta (θ)	Type of Equilibration	Picture of Equilibration Process over Time
$\theta < -0.02$ $\theta = -0.03$	Oscillatory Divergence	
$\theta = -0.02$	Uniform Oscillation	
$-0.01 > \theta > -0.02$ $\theta = -0.015$	Oscillatory Convergence	
$\theta = -0.01$	Instantaneous Convergence	
$0 > \theta > -0.01$ $\theta = -0.005$	Non-Oscillatory (Direct) Convergence	
$\theta = 0$	No Movement	
$\theta > 0$ $\theta = 0.1$	Non-Oscillatory (Direct) Divergence	

Economists characterize the equilibrium solution (if it exists) by describing it either as stable or unstable.

A **stable equilibrium** is an equilibrium position to which the system will return if there's a slight change in the environment. For example, look at the case where $-0.01 > \theta > -0.02$. In this case, if for some reason price were to be pushed above equilibrium, the forces in the system would push price back to the equilibrium.

An **unstable equilibrium** is an equilibrium to which the system will not return if there's a slight change in the environment. For example, consider the last case ($\theta > 0$). Here, if the price starts at the equilibrium value, it will stay there. However, any slight change in price cause the system to zoom away from equilibrium.

An equilibrium is stable if the equilibration process is **convergent**; it is unstable if the equilibration process is **divergent**.

Although the issue of stability is of primary importance, economists are also interested in two other questions about the equilibration process.

If the system converges to equilibrium, we often want to know if it does so "directly" or through an "over-and-under" (or oscillatory) fashion.

Finally, economists also study the system's speed of equilibration. How many time periods does it take to reach equilibrium?

PHASE DIAGRAMS:

Phase diagrams are another tool that we can use to determine the type of equilibration process and the equilibrium solution.

In a phase diagram **we graph $y(t+1)$ as a function of $y(t)$** . We use a **line of slope +1 which passes through the origin** to help us see how the time path will evolve. The slope of the phase line quickly reveals the equilibrium solution(s) and type of equilibration process. Phase lines answer all three questions about the equilibration process in an instant.

Phase diagrams are a nifty but different type of graph. They require you to mentally plot out the movement of a variable over time. They take a little getting used to, but once you understand how they work, phase diagrams are a powerful way to quickly analyze an equilibration process.

Aside on the t notation:

One common source of confusion is with the notation. $y(t+1)$ means the value of y in time period $t+1$. If you are comparing $y(t)$ to $y(t+1)$ and $y(t+2)$, $y(t+1)$ means one time period ahead and $y(t+2)$ means two time periods ahead. Of course, $y(t-3)$ would mean three time periods back.

Example: Suppose we have a time series of GDP like this:

Time	GDP (billion \$)
1990	5522
1991	5678
1992	5951

Then, if we pick 1991 as the t 'th time period, we can identify $GDP(t)$ as \$5678 billion; $GDP(t-1)$ as \$5522 billion and $GDP(t+1)$ as \$5951 billion. The notation of $(t+1)$ in $y(t+1)$ is simply used to indicate which time period you are talking about.

Difference equations (“equations of motion”) of an equilibrium system can be described graphically by showing the movement of an endogenous variable over time (e.g., $Price=f(\text{time})$ or $Profit=f(\text{time})$) or graphing one endogenous variable against another (e.g., $Profit=f(\text{Price})$). In the last chapter, you had some practice drawing a few pictures of the behavior of price over time as θ , the parameter that controlled the equilibrium process, varied.

Another way of graphically depicting the behavior of a variable over time is by using a **phase diagram**. With such a graph, the “result” variable of the system (market price in the last lab) at one point in time is plotted against its value at another point in time. The resulting curve or line is called a **phase line**.

Why Study Phase Diagrams:

The phase diagram is not just another pretty picture!

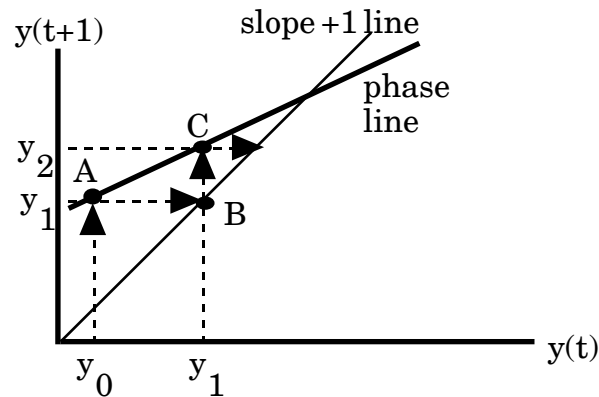
The **LESSON OF PHASE DIAGRAMS** is that

*Different equilibration processes produce different phase lines.
This means that you can use a phase diagram to describe the particular equilibration process in which you are interested.*

We will demonstrate to you in this lab that by comparing the slope of the phase line to a line of slope +1 (i.e., a 45° line), **both the equilibrium solution(s) and type of equilibration process (including its speed) are immediately revealed.** This is a powerful aid to the study of the equilibrium process because the phase diagram tells you immediately whether the equilibrium is stable, how it is reached, and how fast it is reached.

Both the sign (positive or negative) and magnitude of the slope of the phase line reveal information about the equilibration process.

How to Read a Phase Diagram:



Phase lines are different from regular lines and must be interpreted in a special way. In the diagram above, there are two lines: a phase line (the dark one) and a line with a slope of +1 which passes through the origin. The phase line represents a specific difference equation that mathematically describes an equilibration process $y_{t+1} = f(y_t)$. Given an arbitrary, initial value, y_0 (plotted on the horizontal axis), the phase line traces out by iteration (i.e., repeated calculation) all of the subsequent values (i.e., $t=1, 2, 3 \dots$) of y .

WARNING! You can't just plow through this reading. Slowly, carefully trace out the movement from A to B to C as you read the explanation below. Use your pen to follow on the graph what the text is describing.

First, since the phase line maps any value of y into its value the next period, we can find $y_1 = f(y_0)$ by going straight up from y_0 to the phase line, hitting point A, and reading the height on the vertical axis as the value of y_1 .

So, given any arbitrary, initial y_0 , we can find y_1 by going straight up to the phase line.

Next, we seek the value of y_2 given y_1 . This requires that we first plot y_1 on the horizontal axis (just like we did when we started with y_0). This is easily done by using the "slope of +1 line" which is the locus of points with identical x and y values, such as (1,1), (8,8) and (12.327, 12.327)). Thus, to

transplot y_1 from the vertical to the horizontal axis, we move across to the “slope of +1 line,” hitting point B, and then turn straight down to the horizontal axis to locate the point y_1 .

In other words, the slope of +1 line allows you to “reflect” the value of y_1 from the vertical axis to the horizontal axis.

We can then find $y_2 = f(y_1)$ by going straight up from y_1 to the phase line, hitting point C, and reading the height on the vertical axis as the value of y_2 . We can trace out the rest of the time path by repeating the above procedure.

Always read phase lines in this way:

From any arbitrary, initial value at $t=0$, go to the phase line to find the next period's value. Then use the “slope of +1 line” (the 45° line) to transplot y axis values to the x axis in order to begin the procedure again.

Properties of the Phase Line:

Now that the nature of the iteration inherent in a phase diagram is clear, we turn our attention to the powerful properties of the phase diagram.

Equilibrium:

The equilibrium value or values (sure, there may be more than one equilibrium point!) of a system are instantly displayed in a phase diagram by ***the intersection (or intersections) between the phase and 45 degree lines.***

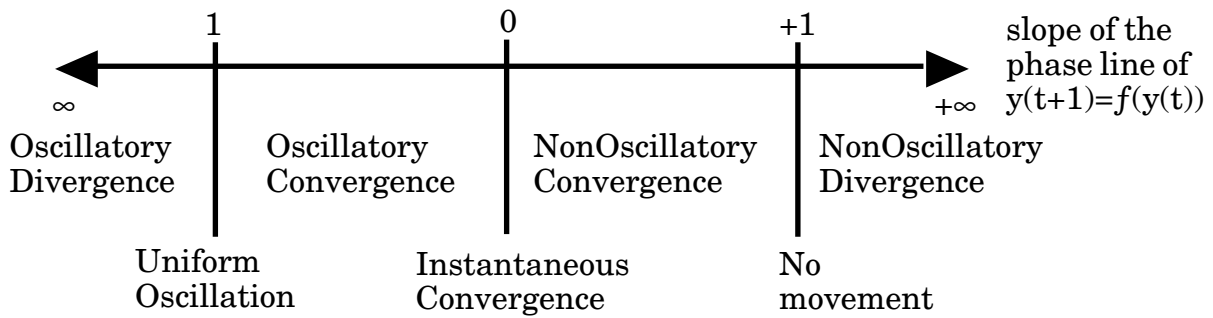
This has to be so, since a point that is simultaneously on the phase line and slope of +1 line will **map a y_t into a y_{t+1} of identical value**; and when $y_t = y_{t+1} = y_{t+2} = \dots$, by definition, we are in equilibrium.

The Slope of the Phase Line:

Although finding equilibrium is important, we are also interested in how an equilibrium might be reached (if at all) given a disequilibrium starting point. For this question the phase diagram is particularly useful: the slope of the phase line at equilibrium (that is, where the phase line crosses the 45 degree line), reveals information about the type of equilibration process and its speed.

For example, the phase line above has a positive slope, but it is less than one (this is obvious since it is flatter than the slope +1 line). Every phase line with $0 < \text{slope} < 1$ exhibits non-oscillatory (or direct)convergence.

In fact, the slope of the phase line can be broken down into seven cases:¹



Not only does the slope of the phase line tell us about the **type** of equilibration process, but in addition its magnitude reveals something about the **speed** of equilibration. For example, a phase line with slope 0.9 will converge (in a non-oscillatory manner, of course) slower to equilibrium than a phase line with slope 0.1. You will establish a pattern regarding speed later in this lab.

The basic rule that emerges from the phase diagram is that the *algebraic sign* of the slope of the phase line (of $y_{t+1} = f(y_t)$) determines whether there will be *oscillation*; while the *absolute value* of the slope of the phase line (compared to 1) governs the question of *convergence*. Along with this basic rule about the type of equilibrium, we have learned that the slope also reveals the speed of equilibration. As the slope approaches zero, equilibrium is reached faster.

Other Phase Lines:

Higher order phase lines, for example, $y_{t+2} = f(y_t)$, also reveal information about the equilibration process. Unfortunately, these are beyond our scope at this time. Before we return to the C15Lab.xls workbook, however, let's make sure we know how to read a phase diagram.

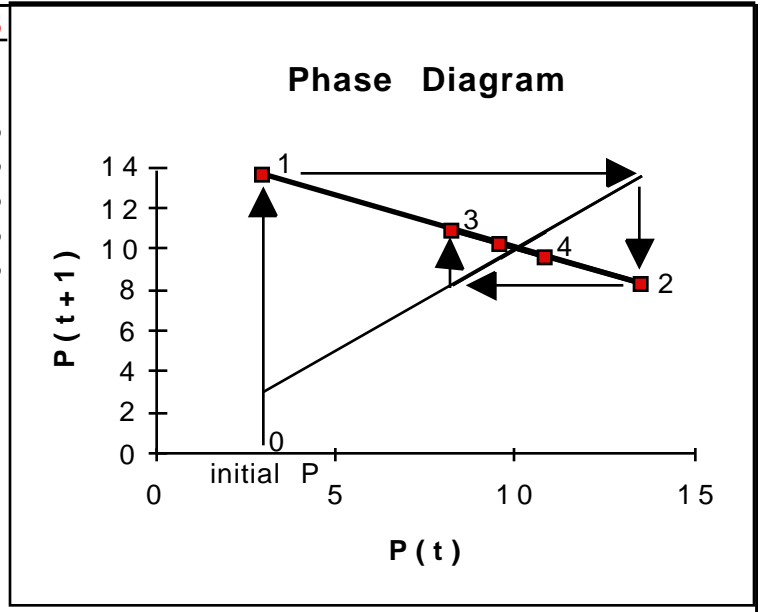
¹ "Hey," you might say, "we found seven cases in our work in the C14Lab.xls file!" Exactly! There is an important tie here. We have to nail this down.

Reading a Phase Diagram of the Profit Equilibration Model:

We now turn to an example of a phase diagram and the interpretation of the slope of the phase line for the profit equilibration model.

Time	P (t)	theta	P (t+1)	Slope +1 Line
0	3	-0.015	13.5	3
1	13.5		8.25	13.5
2	8.25		10.875	8.25
3	10.875		9.5625	10.875
4	9.5625		10.21875	9.5625
5	10.21875		0	10.21875

The phase line has a slope between 0 and -1 (at the intersection with the slope of +1 line). The resulting equilibration process is a "cobweb" and is indicative of oscillatory convergence. As the slope approaches 0 (from -1), the phase line gets flatter and creates a tighter cobweb. Thus, it reaches equilibrium faster.



Notice how we follow the same steps in reading the phase line.

First, we take $P_0=3$, and then go to the phase line which yields $P_1=13.5$.

[Follow along in the table and phase diagram at left. Use your pen to track how it works.]

Then, we go to the line of slope +1 (to transplot 13.5 from the vertical to the horizontal axis).

From here, we go back to the phase line (to $P_2=8.25$). We continue this procedure, i.e., iterate, until we are done.

So, we go *up and down to the phase line and right and left to the line of slope +1*.

Does this make sense?

Begin C15Lab.xls now to create a phase diagram. Good luck!

Speed of Equilibration

Thus far . . .

We have seen that phase lines not only immediately reveal the equilibrium solution(s) (by the intersection(s) of the phase line and slope of +1 Line), but that they also show the type of equilibration (by the sign of the slope of the phase line at the intersection point). If the phase line is positively sloped, but less than one, you immediately know that this is a case of non-oscillatory convergence. If the phase line is negatively sloped, but less than one (in absolute value), you immediately know you are dealing with oscillatory convergence. The other 5 cases (2 ranges of oscillatory and non-oscillatory divergence and three points of uniform oscillation, instantaneous convergence, and no movement) are also immediately revealed by the slope of the phase line.

In addition . . .

The magnitude of the slope of the phase line, i.e., its actual, numerical value, also provides information about the equilibration process. It reveals the speed of equilibration.

As the slope gets closer to 0, from either direction, the convergence toward an equilibrium solution occurs faster. Once the slope is greater than one in absolute value, as the slope gets larger in absolute value, the divergence away from an equilibrium solution gets faster.

Return to C15Lab.xls and complete the assigned task in the **Speed** sheet. You will show that the speed of equilibration does depend on θ and that you can quickly gauge the speed of equilibration by noting the magnitude of the slope of the phase line.

Lessons

To sum up, the equilibration process involves three basic questions which can be phrased in different ways:

1) Is the equilibrium stable?

In other words, stable versus unstable equilibrium?

If the system starts out away from equilibrium, will it return? If it is displaced from equilibrium by a shock, will it return?

2) If it is stable, how will it get to equilibrium?

In other words, oscillatory versus non-oscillatory equilibrium?

Does the system settle down by overshooting the equilibrium, but by less every time — like a swinging pendulum? Or, does it gradually go directly to the equilibrium value without ever overshooting?

3) If it is stable, how fast does it get to equilibrium?

In other words, fast or slow equilibration?

Does the system settle down quickly or slowly?

The answers to these questions can be found by tracing out the time path of the **endogenous variables**. Different types of graphs can be used to answer questions about the equilibration process:

1) An endogenous variable over time: $P = f(t)$.

2) An endogenous variable as a function of another endogenous variable: $\pi = f(P)$.

3) Phase diagram — an endogenous variable as a function of itself at a different point in time:
 $P(t+1) = f(P(t))$.

Of the three kinds of graphs described above, the phase diagram has a big advantage over the other two. What is it? Read on to confirm your answer.

PHASE DIAGRAMS:

In today's chapter we have attacked the above three questions using phase diagrams. The big advantage of phase diagrams lies in their consistency. Different equilibrium models (from profit equilibration to macroeconomics models) will show different types of pictures when the endogenous variable over time or one endogenous variable as a function of another endogenous variable graphs are used. They'll have different "theta" break points for the seven regions. The phase diagram, on the other hand, remains the same. Phase lines with slopes between zero and one will exhibit direct convergence; while a phase line with a slope between zero and negative one has an oscillatory convergent equilibration process.


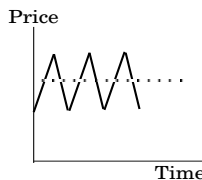
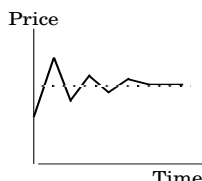
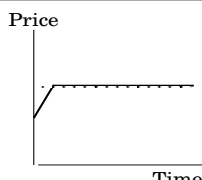
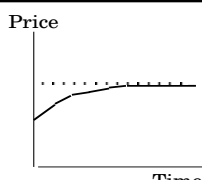
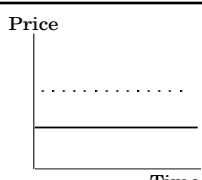
The LESSON and BIG ADVANTAGE of phase diagrams is that the intersection of the phase line with the slope of +1 line shows the equilibrium solution(s) AND that the SIGN (> or < 0) and MAGNITUDE (numerical value) immediately reveal the type and the speed of the equilibration process.

In this lab, you have discovered a precise, consistent relationship between the slope of the phase line and the type of equilibration (including whether the equilibrium is stable, how it is reached if stable, how fast it is reached if stable).

The relationship between the slope of the phase line and the type of equilibration is worth remembering and one that we will use in other equilibrium models.

NAME _____

C15Lab Answers

Value of Theta (θ)	Type of Equilibration	Picture of Equilibration Process over Time	Value of Slope of Phase Line	Picture of Equilibration Process with Phase Line
$\theta < -0.02$ $\theta = -0.03$	<u>Oscillatory Divergence</u>			
$\theta = -0.02$	<u>Uniform Oscillation</u>			
$-0.01 > \theta > -0.02$ $\theta = -0.015$	<u>Oscillatory Convergence</u>			
$\theta = -0.01$	<u>Instantaneous Convergence</u>			
$0 > \theta > -0.01$ $\theta = -0.005$	<u>Non-Oscillatory (Direct) Convergence</u>			
$\theta = 0$	<u>No Movement</u>			
$\theta > 0$ $\theta = 0.1$	<u>Non-Oscillatory (Direct) Divergence</u>	