Chapter 3: Single Variable Unconstrained Optimization

This handout is to be read in conjunction with the Excel file, C3Lab.xls. We will tell you when you are to move from the handout to the Excel file and vice versa. For now, start by reading the handout.

PURPOSE OF THIS CHAPTER READING:

- To outline the direct and the marginal methods for solving simple optimization problems.

- To continue to gain confidence and mastery of Excel, including: cell references, formulas, and charting.

INTRODUCTION TO CHAPTER 3:

Although this chapter contains only one assignment, we recommend that if you are having problems with Excel (formatting cells, relative references, drawing charts), then you should **practice** on your own. Open a new spreadsheet and explore the different buttons and how they work. Be adventurous!

In this chapter, we will be learning about SINGLE-VARIABLE UNCONSTRAINED OPTIMIZATION. You have read "An Overall View to the Economic Approach" (in Chapter 1) so here we simply remind you that

optimization problems
and
equilibrium systems

are the two main types of models that economists construct as they attempt to understand observed behavior.

Recall that within the optimization category, we find

unconstrained
and
constrained

optimization problems.

Unconstrained simply means that the choice variable can take on any value—there are no restrictions. *Constrained* means that the choice variable can only take on certain values within a larger range.

> In a sense, nearly all economic problems are constrained because we are interested almost exclusively in non-negative numbers for variables like output and prices. However, this constraint is so common that we tend to ignore it and think of constrained problems as those with constraints in addition to non-negativity. For example, the budget constraint in the simple consumer problem (remember this? Income must be greater than or equal to Expenditure) restricts the number of goods that you can buy given income and prices. So, we don't want you to think of problems that only restrict values of the choice variable to non-negative numbers as constrained problems.

Within the *unconstrained* optimization problem heading, we can have

single-variable
and
multi-variable

types of problems.

Single-variable problems involve <u>only one</u> choice variable; *multi-variable* problems involve <u>more than one</u> choice variable.

> An example of a single-variable optimization problem is a perfectly competitive firm whose job it is to choose the level of output to maximize profits given the market price and given its cost conditions.
>
> An example of a multi-variable optimization problem is a consumer whose job it is to choose utility-maximizing quantities of beer and pizza given her income and prices.

In this chapter, we will be analyzing the simplest type of optimization problem wherein a single, unconstrained choice variable is chosen by the optimizing decision-maker. We will find the INITIAL solution to this problem by the **direct method** and the **method of marginalism**. Of course, as you well know, this is merely the first step in a four-step process (INITIAL, SHOCK, NEW, COMPARE) called **comparative statics** that forms the heart of the Economic Approach.

All of this is captured on p. 21 in the Road Map in the C1Read.pdf handout. It is important that you know where we are in the grand scheme of things as we work our way through the mathematics and Excel software.

If you haven't done so already, open C3Lab.xls and start reading the file.

SINGLE-VARIABLE, UNCONSTRAINED OPTIMIZATION: A CONCRETE PROBLEM

We will use the following problem as an example of single-variable unconstrained optimization: Suppose that the manager of a firm wants to find the level of output (Q) that maximizes profits (_), defined of course as Revenues (price times quantity) less Total Costs. The price per unit is $4; and total cost in this case can be found by squaring the number of units of output produced. Mathematically, this problem can be expressed

$$\max_{Q} \_ = \$4Q - Q^2$$

> NOTE: An Excel formula for this equation would be = 4*Q - Q^2
> The "^" (SHIFT-6) is the "raise to the power" symbol.

We call this problem a "concrete" problem because the values of the exogenous variables are explicit numbers (e.g., price is $4). Later, we will see "general" problems where the exogenous variables are simply represented by letters (like P or Q^a).

If you haven't done so already, open C3Lab.xls and start reading the file. Read the Introduction sheet, then continue reading below.

STEPS IN SOLVING THE PROBLEM — I) SETTING UP THE PROBLEM:

The set up for this optimization problem, like ALL optimization problems, is composed of three parts:

1) Goal (or objective): to maximize profits

π is the objective function

2) Endogenous variables: the quantity produced

The decision maker is free to choose any value of Q: 0, 16.8734, 222.87 million. (Mathematically, she could even choose negative values, but this makes little economic sense—what is a negative output?)

3) Exogenous variables: the price of $4/unit; the functional form and, the exponent"2" in the cost function

The decision maker CANNOT change these values; that's what we mean by exogenous.

Keep in mind that this example is a **concrete** problem because the values of the exogenous variables are explicit numbers, P=$4 and a=2. A general formulation of this problem, which we will examine later would have us maximize $\pi = PQ - Q^a$.

THE DIRECT METHOD FOR SOLVING THE PROBLEM

Go back to the workbook at Q&A: Screen 1, where you will learn about the Direct Method and answer some questions. Then, come back here.

THE METHOD OF MARGINALISM

You have just solved an optimization problem using the Direct Method, which consists of creating a table or drawing a graph that shows the objective function as a function of the choice variable, allowing one to see exactly where the optimal point lies. Basically, we scan the entire range of possibilities, then pick the best option.

The Method of Marginalism, on the other hand, employs a different strategy. This method involves looking at changes in the objective function as the choice variable changes. If you are trying to find the maximum value of a function, the basic question is this: Is the objective function increasing in value as I change the choice variable? If the answer is "Yes", then keep on changing the choice variable; if "No", then go in the opposite direction.

Why the Name "Marginalism"?

It's called "marginalism" because "marginal" means "change in" or "additional." So, **marginal revenue** means the change in the revenue generated by a unit change in output, while **marginal cost** means the additional cost incurred per unit increase in output.

Note: This definition does not restrict us to looking only at situations where output increases by exactly one unit. We can examine changes in output of any amount, but if the change in output we observe is not equal to one unit, the marginal revenue is still reported as the increase in revenue *per unit of output*. If a firm's revenues increase by $2000 when its output increases by 100, then its marginal revenue is $20. If a firm's revenues increase by $300 when its output increases by 0.2, then its marginal revenue is $1500. Marginal revenue is a ratio, measured in dollars per unit of output. The same goes for marginal cost.

Question: Define '**marginal profit**'. Think for a moment and then read on.

Do not confuse "marginal" in economics with "mediocre" or "barely making it." This is an example of how economics can be difficult because it takes words that have meaning in everyday conversation and gives them a precise, special meaning in the language of economics. You have to build your economics vocabulary and remember the economic meaning of words. "Efficiency" and "elasticity" are two more examples of terms that have a special meaning in economics.

Direct Method versus Marginalism: A Simple Story to Remember

Suppose you had to find the top of a mountain.

> One way would be to get in a plane, fly over it, and spot the peak. You would be able to see the entire mountain and so finding the highest point would be a snap. That's the Direct Method of solving optimization problems.
>
> But what if you were blind? Could you still find the highest point on a mountain? Sure! You could simply use the method of marginalism. Just start walking up the mountain. (Visualize yourself walking up a mountain with your eyes closed.) When you step *up*, keep going in that direction; any time you take a step that takes you *down*, back up and go the other way. You'll know you're at the top when every step you take is down.

Marginalism is nothing more than a strategy for solving optimization problems that relies on tracking the change in the objective function as the choice variable changes. In the mountain example, you track the change in height for each step you take.

Before we continue, the answer to the question on the previous page is:

> Marginal profit is the additional profit generated by one more unit of output.

A Complication: Two Types of Marginalism:

There are two ways to consider changes in a choice variable:

1) ***Discrete*** (or relatively big) changes—denoted mathematically by the symbol "Δ" which means "change in"

2) ***Infinitesimal*** (or vanishingly small) changes—denoted mathematically by the symbol "d" which means "derivative"

This distinction in the kind of change should already be familiar to those of you who understand calculus. If you have not had a calculus course or cannot remember what a "derivative" is, don't worry—we're going to explain it to you.

Let's solve our concrete, single-variable, unconstrained optimization problem first by using the Method of Marginalism with **discrete** changes.

Return to the Excel workbook at Q&A Screen 3.

THE METHOD OF MARGINALISM—INFINITESIMALLY SMALL CHANGES

We've looked at the Method of Marginalism using discrete changes. We now turn to the second type of "change" in the Method of Marginalism, **infinitesimally** small changes.

Marginalism based on infinitesimally small (or "teeny weeny" for the more technical among you) changes means approaching and solving the problem by using calculus.

In order to make the connection, first we must understand what a derivative is. A formal definition is provided below:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Wait, wait, WAIT!!! Don't get scared, it's straightforward and makes a lot of sense, OK? Just give us a chance. The definition says, "f prime of x is defined as the change in f(x) divided by the change in x"—just like we were doing when we constructed those marginal columns (e.g., marginal profit) in the Excel workbook. The only difference from what we were doing before (and it's a **CRUCIAL** difference) is that we keep making the changes in x smaller and smaller and smaller (that's what the "limit as Δx approaches zero" tells us). It's marginalism because it's about **changes**, but it's calculus because it's about infinitesimally small, teensy weensy, tiny miny, itty bitty changes.

In the next part of the Excel workbook, we are going to explore what happens to marginal profit as you make the changes in output smaller and smaller.

Return to the Excel workbook at Q&A Screen 7. Work through all of the questions and then Save your completed workbook.

On the Role of Formulas in Calculus

Having seen how marginal profit at Q=1 goes from $2.20/unit to $2/unit as the change in Q approaches zero, this is an excellent time to comment on the role of formulas in calculus.

Most calculus students know that the derivative of 2x with respect to x is 2. That's an application of the same formula we used in calculating the derivative of $4Q - Q^2$ with respect to Q (which is 4 – 2Q). The general formula is that the derivative of aX^b is baX^{b-1}. In the case of 2x, the exponent on x is 1 so that when you apply the derivative rule you get $1*2x^{1-1} = 2$.

In calculus, there are many other formulas for calculating derivatives. In every case, the formula is a ***shortcut*** to finding the change in the function as the change in x approaches zero. Many students see derivative formulas as painful equations that must be memorized without appreciating how much work the formula saves. Absent a derivative rule, extremely cumbersome numerical approximations are the only alternative. So, remember this: the derivative formula or rule is a shortcut that saves a lot of work!

THINGS TO KNOW AFTER FINISHING CHAPTER 3:

- the difference between endogenous and exogenous variables
- the difference between finding a solution using the direct method and the method of marginalism
- discrete changes and infinitesimally small changes
- how to draw a pretty chart (with labels and everything)
- how derivatives relate to the method of marginalism

CONCLUSION

During the course of this lesson, you have found that in order to maximize _=$4Q - Q^2 we must choose Q*=2, yielding π*=$4. This is the INITIAL solution to this concrete, single variable, unconstrained optimization problem.

The solution can be found either by:

1) *The Direct Method* (using a table or graph of the "Totals" — (Revenue - Cost) or Profit)

2) *Marginalism* (using a table or graph of the "marginals" — MR=MC or M π =0)

We have also discovered that there are two types of marginal changes:

1) Discrete, arbitrarily large changes denoted by Δ

2) Infinitesimally small changes denoted by "d"

Don't be confused: ***in no sense is Δ better than d or vice versa***. They are just different sizes of changes. You can say that Δ approaches d as Δ gets smaller and smaller. This idea is VERY BASIC to mathematical economics and optimization problems and we will encounter it again and again and again.

In addition, do not be alarmed if things are still a bit murky. The ideas discussed here will be repeated many times throughout the remaining chapters. Each time you see marginalism, you'll understand it a little better, until it finally clicks.