

Chapter 12: Two-Variable Constrained Optimization via Calculus and Excel: A Review Problem

Previously...

In Chapter 10, we used Calculus via the Lagrangean Method to solve constrained optimization problems. We learned that on the way to solving the problem we find λ^* and that we can interpret it as revealing the change in the maximum value function for an infinitesimal change in the exogenous limit. In other words, it is ALWAYS TRUE that

$$\frac{d \text{ objective function}^*}{d \text{ exogenous limit}} = \lambda^*$$

For example, if we're maximizing utility subject to an income constraint, λ^* tells us the additional maximum utility we'd get if we had one more dollar of income (or the marginal utility of money).

In Chapter 11, we used Excel and the Comparative Statics Wizard as an alternative approach to solving constrained optimization problems. By simply adding the constraint in the Solver Dialog Box, we can have Excel find the optimal solution. We learned that:

1) If the maximum value function is non-linear in the exogenous limit (as in the cigar and brandy constrained optimization problem of C11Lab.xls), then

$$\frac{\Delta \text{ objective function}^*}{\Delta \text{ exogenous limit}} \neq \frac{d \text{ objective function}^*}{d \text{ exogenous limit}} = \lambda^*$$

2) We also saw how Excel's Sensitivity Report generated a Lagrange Multiplier that is actually λ^* (or a very close approximation to it).

3) If the maximum value function is linear in the exogenous limit (as in the shelf space constrained optimization problem of C11Exer.xls), then

$$\frac{\Delta \text{ objective function}^*}{\Delta \text{ exogenous limit}} = \frac{d \text{ objective function}^*}{d \text{ exogenous limit}} = \lambda^*$$

In both chapters 10 and 11, we continued along the now familiar Economic Approach Road:

Steps 1 and 2:	Set Up and Solve the Problem
Step 3:	Comparative Statics
	How does the optimal value change when there's a shock?

Today...

In this lab you will learn how to use Excel to solve another constrained optimization problem. A realistic, actual problem! We will draw 3D graphs and 2D contour maps and see how λ^* is a shadow price. Let's get to work!

THE PROBLEM: A GENERAL DESCRIPTION

We are going to examine the constrained optimization problem faced by a two-plant firm that has a legally-imposed restriction on the total number of tons of pollution that it may emit. The firm has to decide how much output to produce in each plant under the condition that the total pollution generated not exceed an amount specified by the government.

Total Revenue and Total Cost

The firm's two plants, OldPlant and NewPlant, produce the exact same product (hence, the output from either plant sells for the same price), but OldPlant has higher production costs and generates more pollution. Suppose that the profit functions of the two plants are described by the two equations below:

$$\begin{aligned}\pi_{\text{Old}} &= PQ_{\text{Old}} - 2Q_{\text{Old}}^2 \\ \pi_{\text{New}} &= PQ_{\text{New}} - Q_{\text{New}}^2\end{aligned}$$

- Total Revenue = Price * Output.
Price is denoted by P.
The actual product produced is exactly the same and sells for the same price.
- Notice, however, that Total Costs for OldPlant are twice as high (because of the “2” in front the Q-squared term) as those for NewPlant for the same level of output.

Pollution

In addition, suppose that the pollution generated by each plant is given by:

$$\text{Poll}_{\text{Old}} = 10Q_{\text{Old}},$$

where Poll_{Old} is measured in tons of gunk, and

$$\text{Poll}_{\text{New}} = Q_{\text{New}},$$

where Poll_{New} is measured in the same tons of gunk.

For the firm as a whole we can write:

$$\text{TotalPollution} = \text{Poll}_{\text{Old}} + \text{Poll}_{\text{New}} = 10Q_{\text{Old}} + Q_{\text{New}}.$$

OldPlant is, relatively speaking, an environmental disaster—it's a ten-times worse polluter at the same level of output as NewPlant.

Solving the Profit Maximization Problem while Ignoring Pollution [UNCONSTRAINED OPTIMIZATION]

Before environmental concerns were an issue, the firm would have completely ignored the pollution costs. It would simply have chosen Q_{Old} and Q_{New} in order to maximize total profits ($\pi_{Old} + \pi_{New}$).

This would be a simple, two variable unconstrained optimization problem. Following the five steps in the calculus recipe, we would find:

$$Q_{Old}^* = P/4$$

$$Q_{New}^* = P/2$$

$$\pi^* = 3P^2/8$$

Can you do this problem???

Underlying Graphs of this Solution:

A picture may help you to visualize this unconstrained problem and the solution. It will also give us a chance to practice some more with Excel's 3D Graphics capability and that nifty top-down perspective that is so commonly used in economics.

Let's make $P=100$. Note that, when $P = 100$, $Q_{Old}^*=25$ and $Q_{New}^*=50$, which yields $\pi^*=\$3750$.

Since total profits are a function of TWO variables, Q_{Old} and Q_{New} , the underlying graph can be either a 3-D plot or a 2D contour map. Let's do both of these.

**LAUNCH EXCEL NOW AND OPEN THE FILE CALLED
C12LAB.XLS. WE PROVIDE SOME GUIDANCE ON WHAT TO
DO TO ANSWER THE QUESTIONS THAT FOLLOW ON THE
NEXT PAGE.**

The first question is:

Draw TWO underlying graphs of the unconstrained profit maximization problem (that ignores pollution) described in the previous page

- The first graph should be a 3D plot and the second a 2D Contour Map.
- Appropriately title the graph, label the axes, and make it clear and easy to see the optimal solution.
- Put the two graphs below this screen.

You should follow the example given in the C11Lab Graphs worksheet.

(1) First you must construct a table analogous to the table in the C11Lab Graphs sheet which is illustrated on the next page. You need to show the objective function at various combinations of the endogenous variables. To do this, you must:

(a) Select appropriate intervals and beginning and endpoints for the endogenous variables. You should begin at 0 and go up to, say 50 in one case and 100 in the other. Notice that in the C11Lab Graphs table, the endogenous variables are Cigars and Brandy, with intervals of 1 in the case of brandy and 2 in the case of cigars.

(b) Use relative and absolute references in the formulas appropriately. Take a good look at the formula for the cell C10. It is:

$$=18*\$A10-3*(\$A10^2) + 20*CS8 -(CS8^2)$$

This formula contains the utility function:

$$\text{utility} = 18B - 3B^2 + 20C - C^2$$

The purpose of the relative and absolute references is to save a great deal of typing of new formulas by making it possible to copy this cell down and then across and still have the correct formula in the cells that are copied into.

As it stands, this formula in cell C10 computes utility based on 1 brandy (the value contained in cell A10) and 2 cigars (the value in cell C8). The absolute reference to \$A (in \$A10) tells Excel to build the formula off of the value in the A column, the value that contains the number of brandies. The absolute reference to \$8 (in C\$8) tells Excel to build the formula off the value in row 8, the value that contains the number of cigars. See the screen shot of the table on page 5 and the C11Lab Graphs sheet to see what we mean.

When the formula in this cell is copied down to the cell below (into cell C11), the first cell referred to in the formula becomes \$A11, while the second cell referred to remains C\$8, because we've moved down to row 11 but remained in column C. This is what we want: the formula in cell C11 should refer to the number of brandies in row 11 (the value 2 given in A11) and the number of cigars in column C (the value 2 given in C8). When the formula in cell C10 is copied right to the cell to the right (into cell D10), the first cell referred to in the formula remains \$A10, while the second cell referred to becomes D\$8, because we've remained in row 10 but moved to column D. This is what we want: the formula in cell D10 should refer to the number of brandies in row 10 (the value 1 given in A10) and the number of cigars in column D (the value 4 given in D8).

	A	B	C	D	E	F	G
1	<i>UNCONSTRAINED TWO VARIABLE OPTIMIZATION</i>						
2	LESSON: It's easier to see the optimal solution (10,3,127) from the top-down view of the CON						
3	max utility = 18B - 3B² + 20C - C²						
4		B* --> 3		C*--> 10		U*--> 127	
5							
6	The underlying table that shows utility (in the middle) for combinations of B and C						
7	Brandies	Cigars					
8		0	2	4	6	8	10
9	0	0	36	64	84	96	100
10	1	15	51	79	99	111	115
11	2	24	60	88	108	120	124
12	3	27	63	91	111	123	127
13	4	24	60	88	108	120	124
14	5	15	51	79	99	111	115

Now back to your problem . . .

After you have created the cells that contain the data by copying the pattern in the C11Lab Graphs sheet, scroll around the data in the table and “climb to the top of the hill.” This is a Direct Method solution to finding the optimal values! Verify for yourself that profits are indeed maximized at \$3750 when $Q_{Old}=25$ and $Q_{New}=50$. If they are not, something's wrong with your formulas!

Suppose you find that cell that contains the maximum value of profits is somewhere in the middle of the data, for example, F48 (although your exact cell location may be different). You are now in the middle of the sheet and want to know what values of Q_{Old} and Q_{New} correspond to these coordinates. In this reading and lab, we offer some instructions on how to use the split panes controls to see, say, rows 1 and 2 and rows 48-66 on the same screen and columns A to D and F through L on the same screen. See the illustration on the next page and a worksheet we inserted into C12Lab.xls called Split Panes. The sole purpose of the latter is to show you a sheet with the panes already split up.

The picture below shows a portion of an Excel spreadsheet window. As you scroll down and right through the data, you will lose the row and column labels to your data. For example, right now, you can see that \$666.00 is the profit for a value of 4 units (of Q_{new}), but you can't see the value of Q_{old} in Column A. As you scroll down, you won't be able to see the values of Q_{new} (2, 3, 4, etc.) either.

To get around this problem, you can remember that column G is $Q_{new}=4$ OR you can furiously scroll up repeatedly up and down OR YOU SPLIT THE WINDOW PANE as the picture below documents:

The screenshot shows an Excel spreadsheet window with a table of data. The table has columns labeled 2, 3, and 4, and rows of numerical values. The spreadsheet is split into two panes. Callouts provide instructions on how to use the scroll bar and split pane controls.

Callout 1 (Scroll Bar): This is the scroll bar control. Put the cursor on it, and when it turns into a two-sided arrow, click and drag right/left to control how much scroll bar and how many sheet names you show.

Callout 2 (Horizontal Split Pane): This is the horizontal split pane control. Put the cursor on it, and when it turns into a two-sided arrow, click and drag up/down to split the window pane.

Callout 3 (Vertical Split Pane): This is the vertical split pane control. Put the cursor on it, and when it turns into a two-sided arrow, click and drag right/left to split the window pane.

	2	3	4
196.00	\$ 291.00	\$ 384.00	\$
0	\$ 389.00	\$ 482.00	\$
0	\$ 483.00	\$ 576.00	\$
0	\$ 573.00	\$ 666.00	\$
0	\$ 659.00	\$ 752.00	\$
0	\$ 741.00	\$ 834.00	\$
0	\$ 819.00	\$ 912.00	\$
0	\$ 893.00	\$ 986.00	\$
0	\$ 963.00	\$ 1,056.00	\$
0	\$ 1,029.00	\$ 1,122.00	\$
0	\$ 1,091.00	\$ 1,184.00	\$
0	\$ 1,149.00	\$ 1,242.00	\$
0	\$ 1,203.00	\$ 1,296.00	\$
0	\$ 1,253.00	\$ 1,346.00	\$
1,204.00	\$ 1,299.00	\$ 1,392.00	\$
1,246.00	\$ 1,341.00	\$ 1,434.00	\$
1,284.00	\$ 1,379.00	\$ 1,472.00	\$
1,320.00	\$ 1,417.00	\$ 1,506.00	\$

Once you've found where profits are maximized in the table, go ahead and draw your graphs.

Solving the Profit-Maximization Problem Taking Into Account Pollution [CONSTRAINED OPTIMIZATION]

Look back at the section on the levels of pollution produced by the two plants. That section tells us that at $Q_{Old}=25$ and $Q_{New}=50$, this firm is pumping out

$$\text{TotalPollution} = 10*25 + 1*50 \text{ tons of gunk}$$

In other words, 300 tons of gunk are being generated. Suppose now that our elected representatives have decided that this is an unacceptable level of pollution.

In the recent past, the government might have slapped effluent restrictions on OldPlant (e.g., Pollution Limit=75) or fines per unit of pollution. The topic of most of the rest of this lab is how firms behave under such restrictions on the amount they can pollute. At the end of the lab, we ask how much firms would be willing to pay to change the restriction. We will pursue this last issue more carefully in Chapters 19 and 20 by considering a more recent approach to pollution control, namely pollution permits.

Pollution Limits

Suppose that the government sets a limit of 150 tons of gunk in pollution for the entire firm (Pollution Limit = 150). What are the profit-maximizing choices of Q_{Old} and Q_{New} now?

Be Careful To Note The Difference Between Pollution Limit And Total Pollution. Pollution Limit is the exogenous value of pollution allowed which constrains the firm; Total Pollution is the total amount of pollution being produced at a given choice of Q_{Old} and Q_{New} .

The crucial feature of this new problem is that it is a TWO-VARIABLE CONSTRAINED OPTIMIZATION PROBLEM.

There are several ways to find the answer to this problem, including what we've called:

1) The Direct Method

Find all of the feasible (that is, allowable) combinations of Q_{Old} and Q_{New} and pick the one with the highest profit. Even with Excel, this could take a while . . .

A SECOND WAY to find the answer to the constrained optimization problem is:

2) Drawing Graphs

We could try to use the Direct Method by drawing a graph. We might then be able to quickly pick out the optimal solution by scanning the entire graph.

Let's try that with Excel.

Go to the C12Lab.xls Con sheet now and create the graph.

Return here and go to the next page when you are done in Excel.

A THIRD WAY to find the answer to the constrained optimization problem is:

3) The Method of Marginalism via Discrete Change (Δ)

Pick any feasible point and move to another feasible point. If profits increased, continue; if not, go in the other direction.

$Q_{Old} = 10$ and $Q_{New} = 50$, then $TotPoll = 150$ so that's feasible, and $\pi = \$ 3,300$.

$Q_{Old} = 11$ and $Q_{New} = 40$, then $TotPoll = 150$ so that's feasible, and $\pi = \$ 3,258$.

Back up and head the other way. This could take a while . . .

A FOURTH WAY to find the answer to the constrained optimization problem is:

4) The Method of Marginalism via Calculus (d)

This relies on a smarter version of the previous scheme. Since we know the optimal solution is where there's no change in profits, we take derivatives (which measure the change in profits for infinitesimally small changes in Q_{Old} and Q_{New}) and set them equal to zero. We can then solve for the optimal values.

The complication is the constraint. We get around this by using Lagrange's Multiplier Method. We apply the five steps as follows:

As you read this, you should be asking yourself, If handed a problem like this one, could I solve it?

Step 1) Write the problem mathematically:

$$\max_{Q_{Old}, Q_{New}, \lambda} L = PQ_{Old} - 2Q_{Old}^2 + PQ_{New} - Q_{New}^2 + \lambda(\text{Pollution Limit} - 10Q_{Old} - Q_{New})$$

Step 2) Take the partial derivatives with respect to the endogenous variables:

$$\frac{dL}{dQ_{Old}} = P - 4Q_{Old} - 10\lambda$$

$$\frac{dL}{dQ_{New}} = P - 2Q_{New} - \lambda$$

$$\frac{dL}{d\lambda} = \text{Pollution Limit} - 10Q_{Old} - Q_{New}$$

Step 3) Set the partial derivatives equal to zero:

$$P - 4Q_{\text{Old}}^* - 10\lambda^* = 0$$

$$P - 2Q_{\text{New}}^* - \lambda^* = 0$$

$$\text{Pollution Limit} - 10Q_{\text{Old}}^* - Q_{\text{New}}^* = 0$$

Step 4) Solve for the optimal value of the endogenous variables:

After some algebra, you will get,

$$Q_{\text{Old}}^* = \frac{5}{51} \text{Pollution Limit} - \frac{3}{68} P$$

$$Q_{\text{New}}^* = \frac{1}{51} \text{Pollution Limit} + \frac{15}{34} P$$

$$\lambda^* = \frac{2}{17} P - \frac{2}{51} \text{Pollution Limit}$$

Step 5) Find the maximum value function, $\pi^*(Q_{\text{Old}}^*, Q_{\text{New}}^*, \lambda^*)$ in terms of the exogenous variables.

OK, we admit that this is truly ugly. After some painful algebra we simplify to:

$$\pi^* = \frac{1}{17} \cdot \left(\frac{27}{8} P^2 + 2 \cdot P \cdot \text{Pollution Limit} - \frac{\text{Pollution Limit}^2}{3} \right)$$

A FIFTH WAY to find the answer to the constrained optimization problem is:

5) Use Excel's Solver

Last, we can use Excel's Solver to find the optimal solution.

You will do this shortly.

SUMMARY:

Because this is the last chapter on optimization, we take a moment to encourage you to think about the fact that you have learned FIVE DIFFERENT WAYS TO SOLVE OPTIMIZATION PROBLEMS. From graphs, to calculus, to Excel, you are holding a toolkit of solution algorithms. If one way doesn't work, don't hesitate to try another.

Of course, you should also note the similarities between the alternative solution strategies.

And now we turn to the hallmark of the Economic Approach, comparative statics.

Comparative Statics on the Profit-Maximization Problem Taking Into Account Pollution

We've found a solution to the constrained profit-maximization problem faced by a polluting firm that is given some exogenously set pollution limit. The next step, naturally, is to do Comparative Statics.

Instead of the usual Comparative Statics questions which you may find repetitive and boring, we offer some **challenging** questions for you.

The C12Lab.xls Q&A sheet has the same questions that are provided below. Read the questions here and the suggestions that follow before you begin your work in the Q&A sheet. By first reading all of the questions and our suggestions, you will better understand what's going on.

Use the cells that have been set up and defined for you in the Q&A sheet along with Excel's Solver to find the optimal solution when $P=100$ and Pollution Limit=150.

Put your answers to the questions in the 5 text boxes in the Q&A sheet.

Five Questions:

- 1) In the Con sheet, we asked you to find the optimal solution to the constrained problem by scrolling around the data. Here you can use Solver to find a solution. How close were these two methods for you?
- 2) Is the Pollution Limit elasticity of Old Plant Output positive or negative? What does that mean?

A Pollution Permit Scheme

Now suppose the government were to issue pollution permits which could be bought and sold. Each permit gives the firm the right to pollute a ton of gunk. You **MUST** have a permit to pollute one unit and you **MUST** have 67 permits to pollute 67 units. In terms of our discussion thus far,

$$\text{Pollution Limit} = \text{Number of Permits Owned by the Firm}$$

The new wrinkle is that the firm can increase its Pollution Limit by buying additional permits. In other words, the firm can relax its constraint a little by buying another permit. Permits aren't free, however. The firm has to decide if the price at which the permit is being offered is worth it.

So, now that you understand what a pollution permit is, here are two questions:

3) A) If this firm had 150 pollution permits, how much is the maximum it would be willing to pay for one more permit? (We need a \$/permit value.) Explain why. Get a Sensitivity Report for Pollution Limit = 150. Name the sheet.

B) If this firm had 45 pollution permits, how much is the maximum it would be willing to pay for one more permit? (We need a \$/permit value.) Explain why. Get a Sensitivity Report for Pollution Limit = 45. Name the sheet.

4) If this firm had no pollution permits, and the price were \$10/permit, how many permits would it buy? (We need a "number of permits" value.) Explain why.

When you work on Q4, think about these three things:

1) The problem has been radically changed so that there is NO Pollution Limit constraint.

2) The firm can choose how many permits to buy.

3) There is a new exogenous variable, the Price of Permits.

Given the three things listed above, you should change the Solver Parameters dialog box information in order to solve this new optimization problem.

NOTE: Do NOT simply say, "45 permits." DO use Excel's Solver on the new optimization problem to find how many permits the firm will buy at \$10/permit.

5) Given your answers to 3 and 4 above, why is λ^* called a shadow price?

The questions in this lab are difficult and require time and serious thought. We are not leading you by the hand through a series of steps. You decide whether the Comparative Statics Wizard is needed or calculus or whatever. **Good luck!**

Just in case you run into trouble, however, we offer a few suggestions and hints below!

[C12Lab.xls Suggestions and Hints:](#)

- Q2 asks you to explore the effect of a change in the Pollution Limit on optimal Old Plant Output. A student might ask, "'HEY! Exogenous variables are FIXED UNCHANGING CONSTANTS! How can they be changed???"

The answer is, "You have to know where you are in the problem." From the perspective of a decision-making agent solving an optimization problem, exogenous variables are fixed unchanging and unchangeable constants. That's in Step 2 of the Economic Approach: Finding the Optimal Solution.

However, remember that there are THREE steps in the Economic Approach. In Step 3: Comparative Statics, we (the investigators) ask, "How would the optimal value change when there's a shock?" We, the investigators, play a little game with the agent. We change an exogenous variable to a new fixed value, one that is unchanging and unchangeable by the decision-maker. We then ask, "How big was the agent's response? A lot or a little?" That's what elasticity tells us!

So, remember, "You have to know where you are in the problem."

- As you work on Q3, keep this in mind: If a firm had 150 permits, it would be polluting 150 units of gunk and, by optimally allocating the 150 units to Old Plant and New Plant, it would earn some maximum level of profits (which you found in Q1). If it gets one more permit, it will be able to produce one more unit of gunk and, presumably, increase its maximum profits. By how much will profits increase? That's an important question that holds the key to how much the firm is willing to pay for another permit!

If you do Q3 correctly, you should find that the firm is willing to pay MORE for an additional permit when it only has 45 permits compared to when it has 150 permits.

- For Q4, as you redesign the Solver Setup Table and reset the Solver Parameters dialog box, we remind you that Excel may have trouble with named cells in constraints. You may have to delete the name before you can delete a constraint.

If you run into Solver woes, you could just insert a new worksheet (execute Insert: Worksheet) and put in a whole new Solver Setup Table. The Solver parameters "belong to" the worksheet. In other words, each worksheet has its own Solver information.

Lesson: In general, if you find that Excel is confused when using Solver, one quick and easy solution is to insert a new worksheet and start with a clean Solver.

- In Q5, we are asking you to pay attention to the economic interpretation of λ^* . In part B) of Q3, you will find that at Pollution Limit = 45, $\lambda^* = \$10/\text{permit}$. In Q4, you will find that at $\$10/\text{permit}$, the firm buys 45 permits. Wow! What's going on here? Answering that question will help you explain why λ^* is a shadow price!

That's it for suggestions and hints. **Good luck!**