Chapter 10 Exercise: Constrained Optimization

SUGGESTED ANSWERS

Bert collects two things: mathematical economics books (b) and baseball cards (c). Bert's utility function is $U(b,c) = b + 100c - c^2$, where b represents the number of econ books on his shelves, and c is the number of boxes of baseball cards on the shelves. In his office, he has a total of 500 linear inches of shelf space to allocate between these two collections, and he is trying to figure out what the utility maximizing display arrangement is. Books take up one linear inch apiece, and boxes of baseball cards, being slightly larger because each set is complete, take up four linear inches.

Q1: Using calculus, determine the optimal values for b, c, and λ, the Lagrangean multiplier, as well as the maximum value of Bert's utility function. Fractional values are OK.

In your answer, write out the five steps and show your work.

NOTE: The algebra you will need to use in Step 4 (solving the first-order conditions for b^*, c^* and λ^*) is easy in this problem.

Answer to Q1: We can work our way through the five steps in the recipe for solving this constrained optimization problem via calculus:

Step 1) Write the problem mathematically as an <u>unconstrained</u> problem by utilizing the Lagrangean

(a) Write out the constraint in the form Constraint = 0.

The constraint is $500 = b + 4c$ which we rewrite $500 - b - 4c = 0$.
Bert can display 500 books because they only take up 1 linear inch apiece. He can only display 125 boxes of baseball cards, however, because they take up 4 linear inches each.

(b) Multiply the lefthand side of the constraint by λ, to get

$$\lambda \bullet (500 - b - 4c)$$

(c) Add the new term to the objective function to form the Lagrangean Function, L:

$$L(b, c, \lambda) = b + 100c - c^2 + \lambda \bullet (500 - b - 4c)$$

This move transforms the problem from an unsolveable 2 variable, 1 constraint problem, into an easily solved 3 variable, unconstrained optimization problem.

Step 2) Find the **partial derivatives** of the objective function with respect to the endogenous variable**s**.

$$\frac{\partial L}{\partial b} = 1 - \lambda$$

$$\frac{\partial L}{\partial c} = 100 - 2c - 4\lambda$$

$$\frac{\partial L}{\partial \lambda} = 500 - b - 4c$$

Step 3) Set the partial derivatives equal to zero. You should have as many equations as there are endogenous variables. The endogenous variables should all carry asterisks, *, to denote that they are being placed at their optimal values.

$$\frac{\partial L}{\partial b} = 1 - \lambda^* = 0$$

$$\frac{\partial L}{\partial c} = 100 - 2c^* - 4\lambda^* = 0$$

$$\frac{\partial L}{\partial \lambda} = 500 - b^* - 4c^* = 0$$

These three equations are said to be **RELATED** but they look pretty easy to solve.

Step 4) Solve for the optimal values of the endogenous variables. That is, solve **the set of n simultaneous equations (called first-order conditions)**, **one equation for each endogenous variable**, Obviously, $\lambda^*=1$

By substituting $\lambda^*=1$ into $\partial L/\partial c$, we can easily find that $c^*=48$.
Finally, by substituting $c^*=48$ into $\partial L/\partial b$, we get $b^*=308$.

Step 5) The final step is to evaluate the objective function at the optimal values of the endogenous variables:

$U^*(b^*, c^*) = [308] + 100[48] - [48]^2 = 2804$

Q2: Give an interpretation of the value of lambda in this case. You should start with the general fact that "λ^* tells us how much the maximum value function changes as the constraint is relaxed" and say what that means in this particular problem.

Answer: λ^* tells us how much the maximum value function changes as the constraint is relaxed. So, $\lambda^*=1$, in this problem, tells us that if Bert had a teeny bit more shelf space, there would be a unit-fold increase in the maximum utility obtainable.

Since $\lambda^*=1$, U* as a function of shelf space is a straight line and so the size of the relaxation of the constraint doesn't matter. For example, say we increase the shelf space by 1 linear inch to 501 linear inches or by 10 inches to 510 linear inches. Recalculation of the problem will show the optimal solution:

Shelf space (in.)	b* (number of books)	c* (number of card boxes)	λ^* (utils/in.)	U* (utils)
500	308	48	1	2804
501	309	48	1	2805
510	318	48	1	2814

Every increase in shelf space results in an equal increase in U*. That's what $\lambda^*=1$ means.